Option Pricing

Quantitative Finance

2022-2023

TÉCNICO LISBOA

Outline

1. Risk (definition and properties)
2. Options
3. Put-call parity
4. Options strategies
5. Options pricing

TÉCNICO LISBOA

Risk

- A product is risk-free if you know exactly what will be the return of your investment

  Leaving out:

  - Inflation

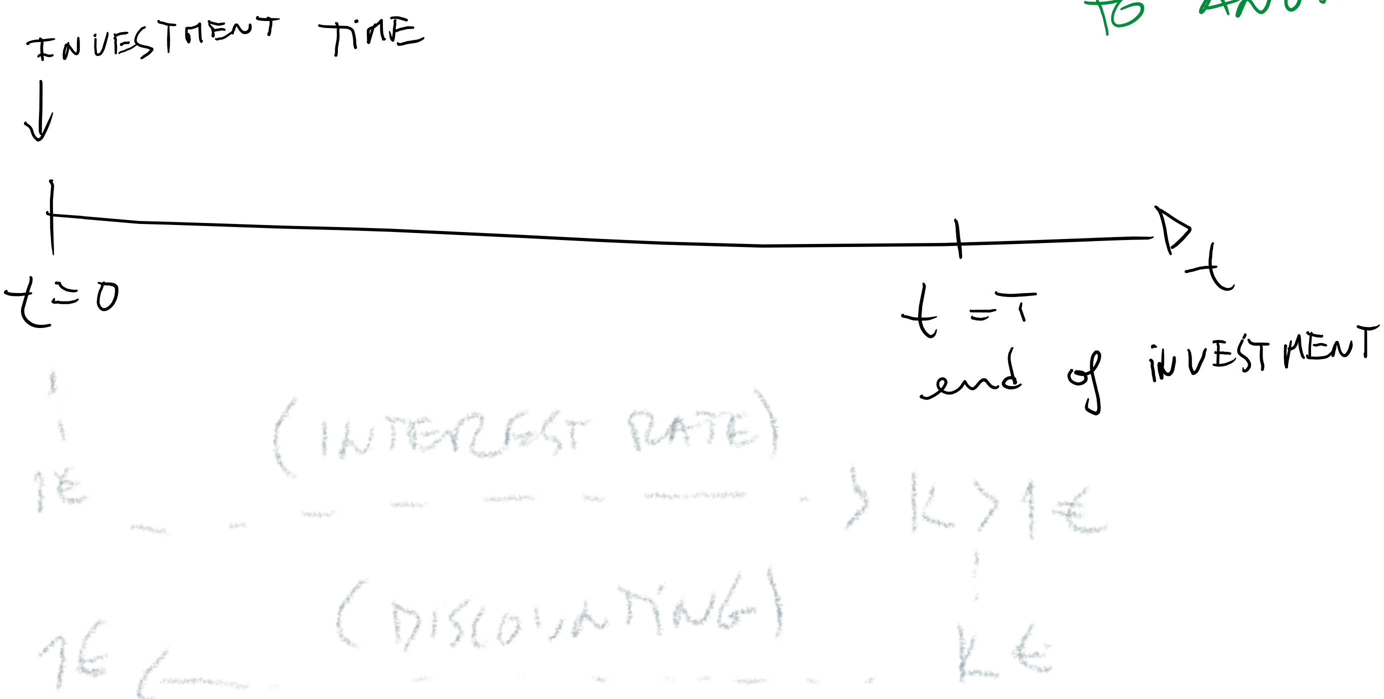
  - interest rate: specially in long term investments the interest rate <u>may not be defined in advance</u>

  This means, in particular, that in reality, the interest rate is random (stochastic process)

Risk-free aplications

1. Deposits in bank
2. Bonds

INTEREST RATES ALLOW US TO MOVE FROM ONE CAPITAL TO ANOTHER



Risk-free aplications

- Therefore we need to update the value of Money, taking into account the interest rates

Interest rates

- Fixed: R (1)
- Time-varying, deterministic:

$\{R(t), t \geq 0\}$ (2)

- Time-varying, stocastic:

$\{R(t), t \geq 0\}$ (3)

With R(t) being a random variable

During the course we assume 1, i.e, the interest rate is fixed and known. We could also assume 2; this would need some "extra" notation

$R(t) = R, \forall t$

with R being known

Interest Rates

Assumption:

- The interest rate(s) are deterministic and fixed along time:

$$R = \{R(t), t \geq 0\} \rightarrow R(t) = R, \quad \forall t$$

Let B(0) be the initial capital (Money at time 0) and R is the annual interest rate

- Annual compounding:

$$B(1) = (1+R)\,B(0)$$

$$B(n) = (1+R)^n\,B(0)$$

- 6-month compounding:

(2 times compounding)

$$B(0.5) = \left(1+\frac{R}{2}\right)B(0)$$

$$B(1) = \left(1+\frac{R}{2}\right)^2 B(0)$$

$$B(n) = \left(1+\frac{R}{2}\right)^{2n} B(0)$$

TÉCNICO LISBOA

Interest Rates

- K times/year compounding:

(K times compounding)

$$B(1) = \left(1 + \frac{R}{K}\right)^{K} B(0)$$

$$B(n) = \left(1 + \frac{R}{K}\right)^{Kn} B(0)$$

- Continously compounding:

$K \to \infty$

$$B(n) = e^{Rn} B(0)$$

$$B(t) = B(0)\, e^{Rt}$$

And this is the most "interest" way of compounding

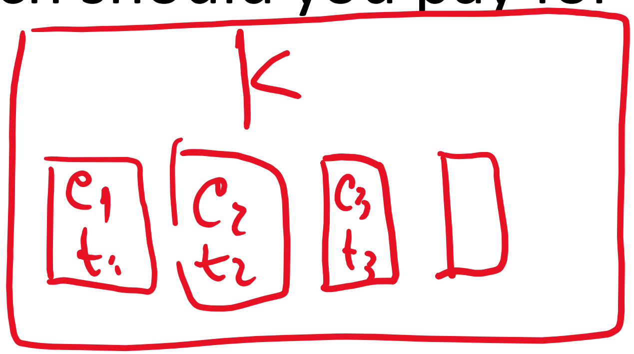
Bonds

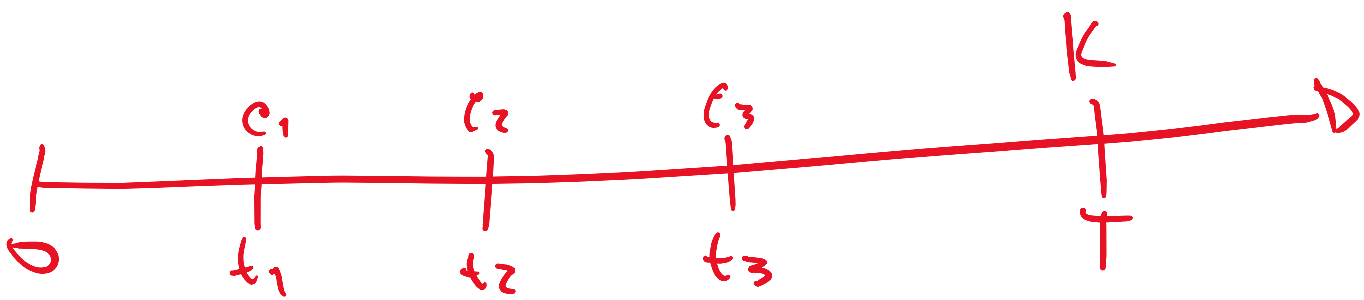
- This is a risk free product, and in fact it has the same "meaning" as deposits or loans
- Bond is a contract between two parties (buyer and the writer $\longrightarrow$ the one that is selling the bond) where they fix:
  - Time T (maturity or time horizon)
  - Face value K (absolute value of the bond, which is the return of the bond at time T)
  - Coupons (if the bond has J coupons each one will have a face value Cj, and a time your are intitled to receive Cj, Tj)

TÉCNICO LISBOA

Bonds

How much should you pay for this contract?

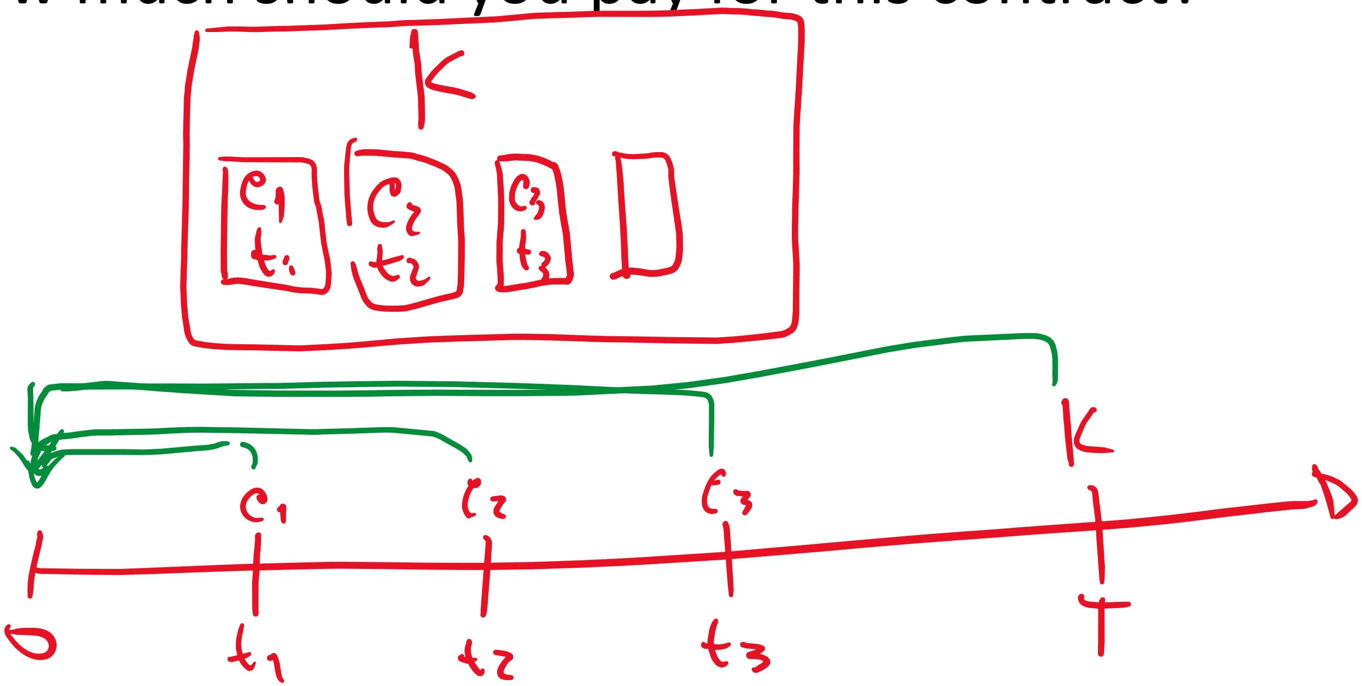




TÉCNICO LISBOA

Bonds

How much should you pay for this contract?



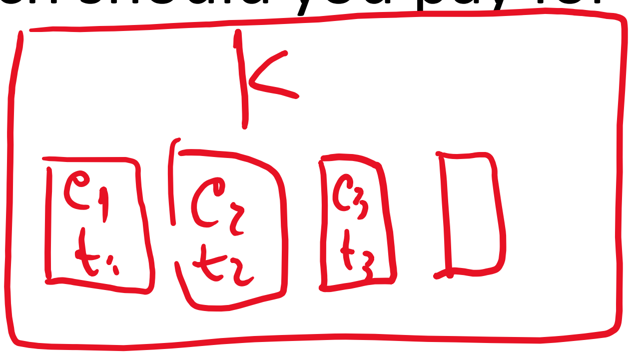
$$\text{price} = c_1 e^{-Rt_1} + c_2 e^{-Rt_2} + c_3 e^{-Rt_3} + K e^{-RT}$$

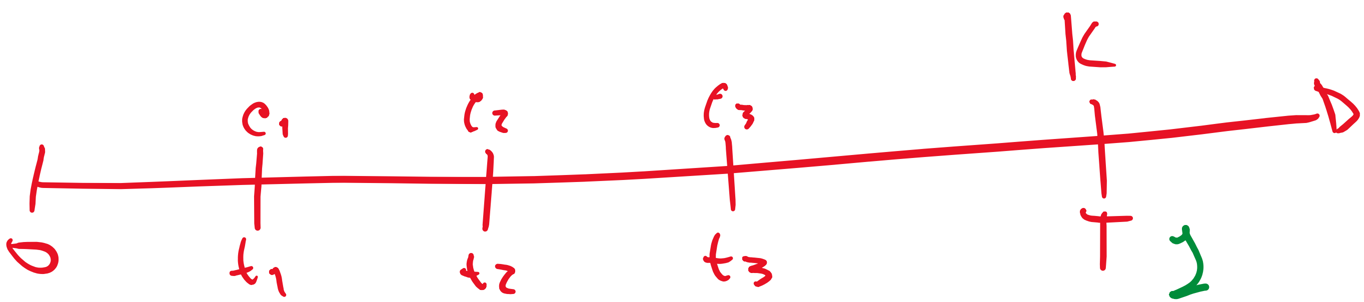
(continuously compounded interest rate)

TÉCNICO LISBOA

Bonds

How much should you pay for this contract?





$$\text{price (in general)} = K e^{-rT} + \sum_{j=1}^{J} c_j e^{-r t_j}$$

Bonds

- For the rest os the course, we will be using mostly zero-coupon bonds $(c_j = 0, \forall j)$

Price of a zero coupon bond = $K e^{-Rt}$

if $\{B(t), t \geq 0\}$ denotes the value of your investment along time,

then:

$$B(t) = B(0) e^{Rt} \Leftrightarrow d B(t) = R dt$$

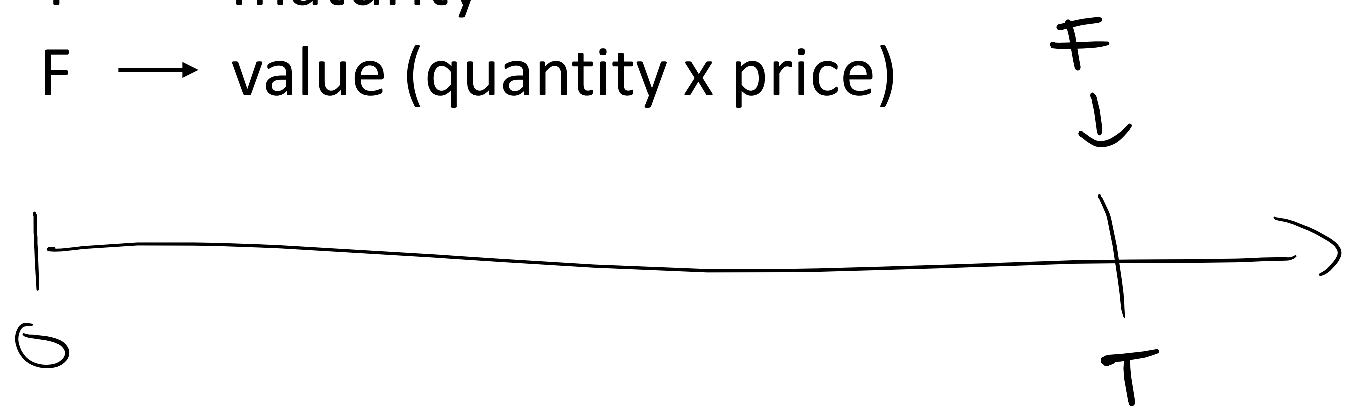
Non Risk-free assets

- Future/forward (they are very usual in the areas of energy, oil, commodities like gold, aluminium....)

Again this is a contract written by two parties (buyer and the seller/writer) where we fix:

T $\longrightarrow$ maturity

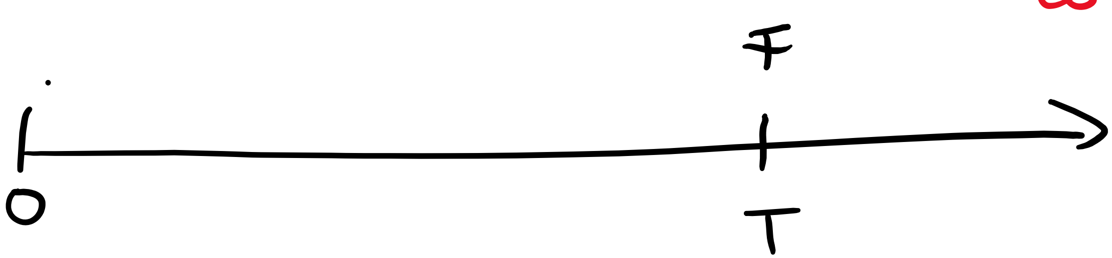
F $\longrightarrow$ value (quantity x price)



Future/forward

- At the maturity, the buyer will pay F (where F represents the amount of commodity agreed in the contract)
- The main diference between <u>Bonds</u> and <u>Futures</u> is that in a bond, there is an initial price (and therefore na initial investment) whereas in the future contract the payment occurs only at the maturity.
- By <u>non-arbitragem arguments,</u> the “fair” price of such contract should be:

$S(t) \to$ price of the commodity at time T



Future/forward

Therefore there should be some relation between F and S(0). In fact:

$$F = S(0)e^{RT}$$

Contracts of some good whose price is changing, stochastically, along time, this expression lacks of some justification

Future and forward contracts won´t be studied in this course

TÉCNICO LISBOA

Non Risk-free assets

Risky products mean that you cannot kown, at time 0, what is the return (if it exist) of your investment:

probability loosing money > 0

probability earning money > 0

Here we will mention two:

- Stocks
- Options

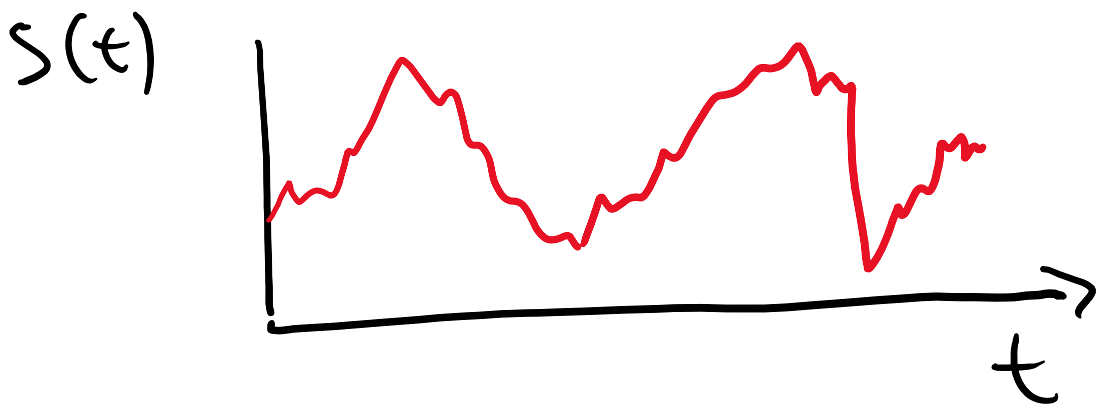
TÉCNICO LISBOA

Non Risk-free assets

Notation: $\{S(t), t \in (0,T)\}$

Where S(t) is the price of a stock at time t. due to a lot os unkonwn factores the prices of stocks change along time, in a random way.

Usually the sotck market has a lot of <u>volatility</u> (standard deviation)

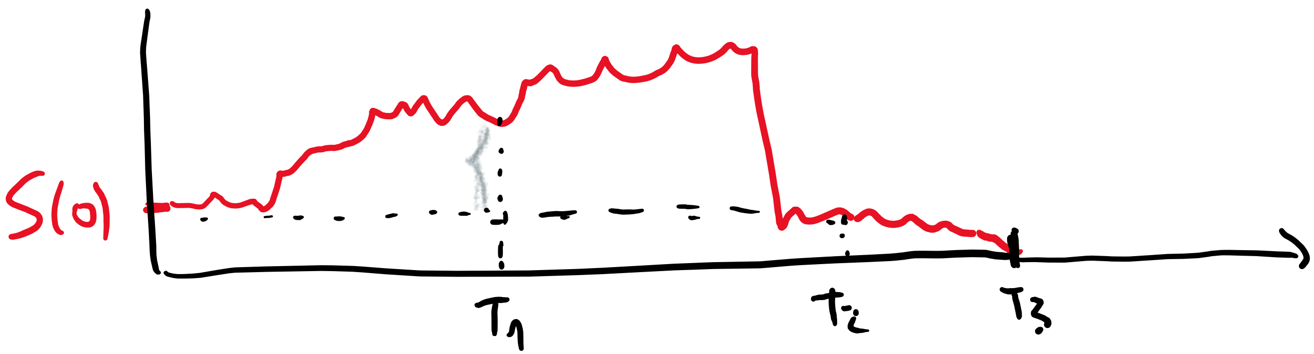


Non Risk-free assets

How do we model this stocastic process?

As the evolution of the stock prices is very difficult to model, we cannot expect to have "precise" forecasts on the future evolution of stock prices.

Investment in stocks is, at least from the set of products that we will discuss in here, the one with higher risk, because you may loose all your money.



Non Risk-free assets

Right now we have two "products"

bonds   stocks

Definition: a portfolio h is a process, where:

$$\{ h(t) = (x(t), y(t)), \ t \geq 0 \}$$

Where x(t) = number of bonds with face value 1 at time t

And y(t) = number of stocks at time t

Non Risk-free assets

In the previous definition, we allow $x(t)$ and $y(t)$ to take negative values. What is the meaning of this?

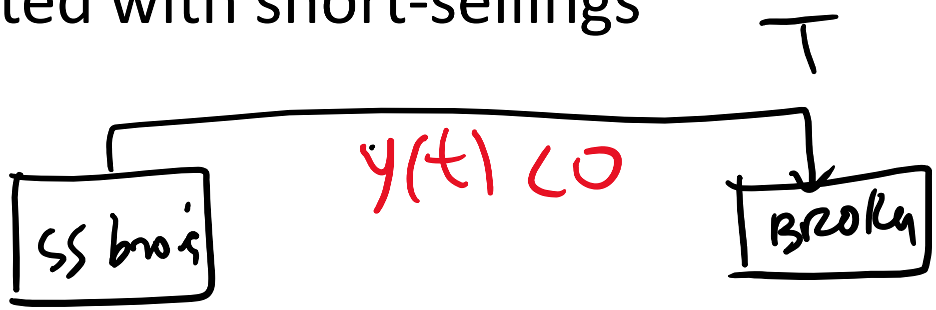
Non Risk-free assets

In the previous definition, we allow x(t) and y(t) to take negative values. What is the meaning of this?

x(t) < 0 means that you have a loan, that you need to return at the maturity

short position on a bond

y(t) < 0 means that you have a short position in a stock. This is related with short-sellings



Non Risk-free assets

Assumptions:

1. Short-selling is possible, without cost
2. You can have a fraction of a stock
3. No transaction costs
4. The market is liquid, meaning that you may buy or sell at anytime, and you find a seller or a buyer.

Definition: the value of a portfolio h, wich we will denote by V^h is the folowing process:

$$V^h = \{ V^h(t) = x(t) + y(t)\,\underset{\sim}{s(t)},\ t \geq 0 \}$$

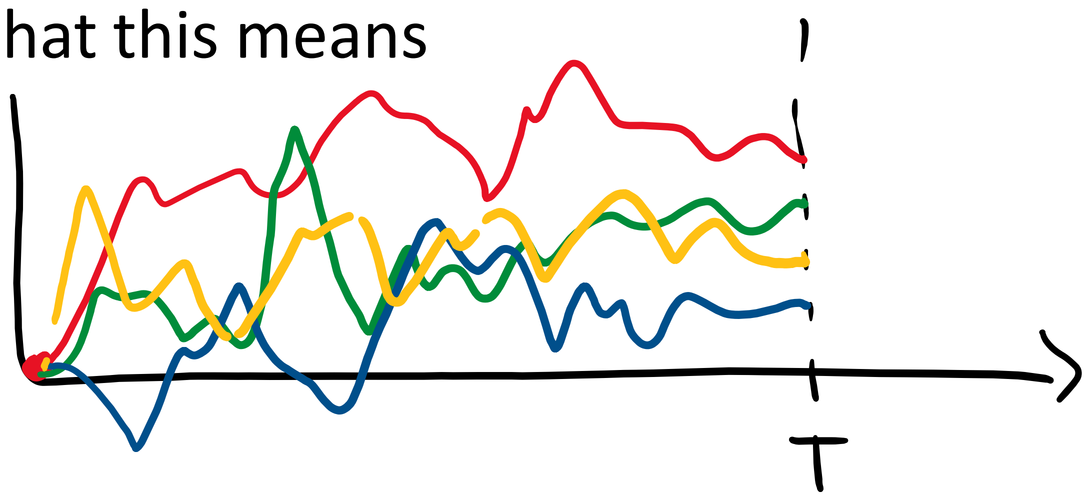
Non-arbitrage

"There are no free lunches"

Definition: an arbitragem portfolio h is a portfolio with maturity T such that the folowing holds:

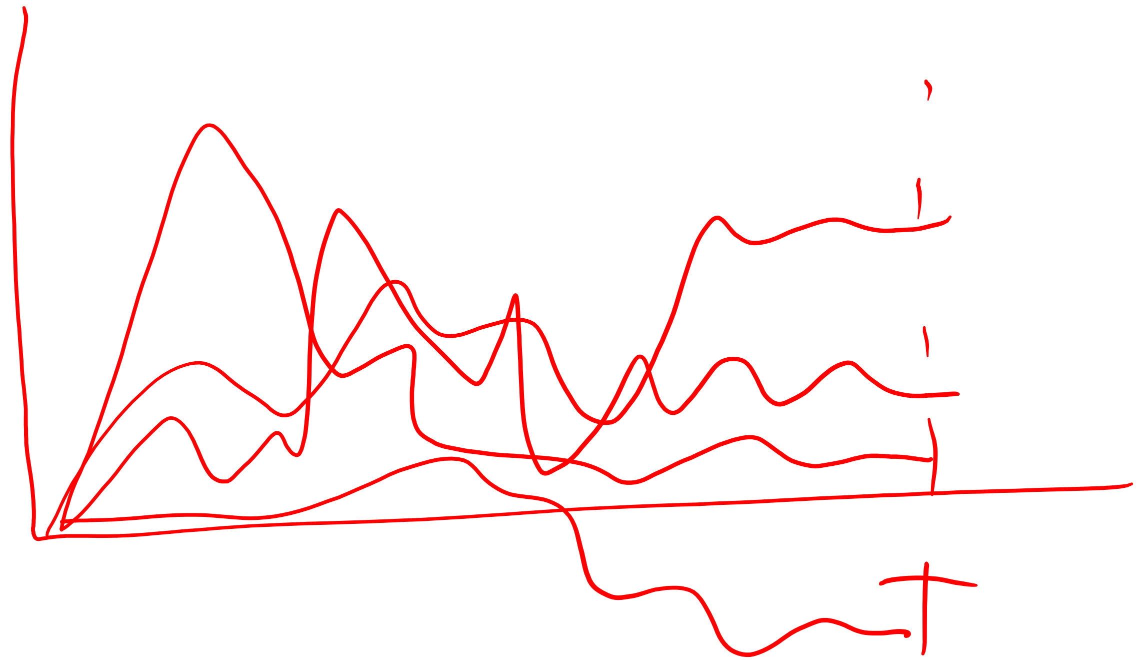
$$P(V^h(t) > 0 \mid V^h(0) = 0) = 1$$

Let's see what this means



Non-arbitrage

Non-arbitrage



From now on we will always assume non-arbitrage portfolios.

Non-arbitrage

What does it mean $V^h(0) = 0$?

$$x(0) + y(0)S(0) = 0 \quad (=)$$

$$y(0) = -\frac{x(0)}{S(0)}$$

Non-arbitrage

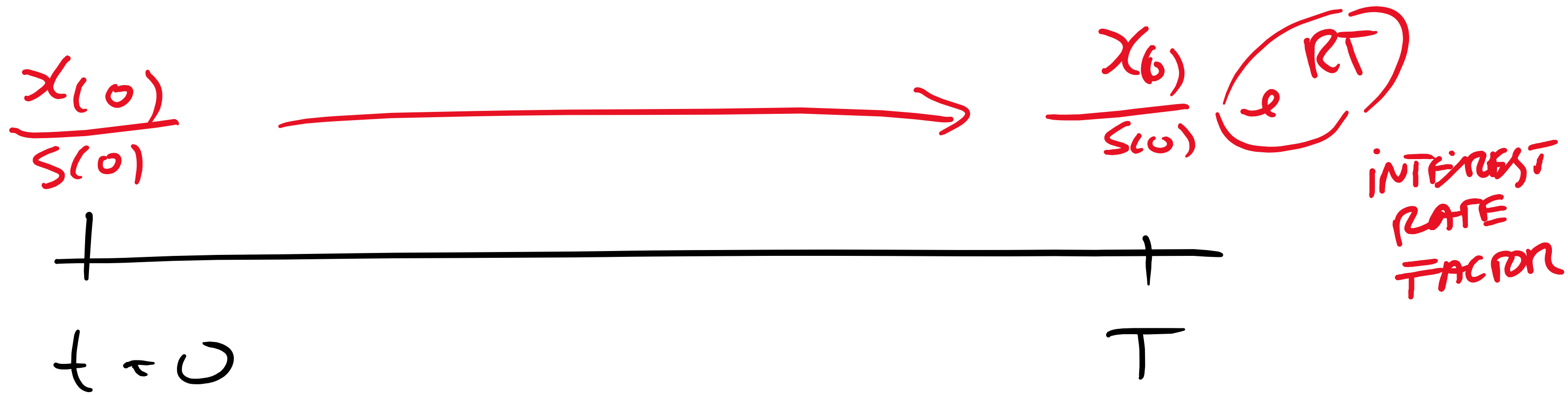
There are two ways to start and investment with zero initial investment:

i. Short selling y(0) stocks, you sell them and invest the money in the bank. From this point on, you do whatever you want. At time T you need to return y(0) stocks (this can be a “wise” strategy if you believe that the stock price is going do decrease)

ii. You borrow x(0)/S(0) from the bank, wich then you use to invest in stocks. From that point on you do whatever you want. At maturity T you need to pay to the bank: $\frac{x(0)}{S(0)} e^{RT}$

TÉCNICO LISBOA

Non-arbitrage



First consequence of non-arbitrage principle:

Options

First consequence of non-arbitrage principle:

Let h1 and h2 be two portfolios written in the same "underlying asset" (these two portfolios depend on the same stock) such that:

$$V^{h_1}(t) = V^{h_2}(t) \quad \text{with probability one}$$

$$\forall \omega \in \Omega : V^{h_1}(t)(\omega) = V^{h_2}(t)(\omega)$$

i.e. for all possible sample-paths, the value of both portfolios at maturity is equal. Then both portfolios will have the same price (non dominance principle).

TÉCNICO LISBOA

Options

Options: There is a contract between two parties (buyer and the seller) where it is fixed:

maturity T

strike price (or exercise price) K

underlying asset (stock...) $\{S(t), t \geq 0\}$

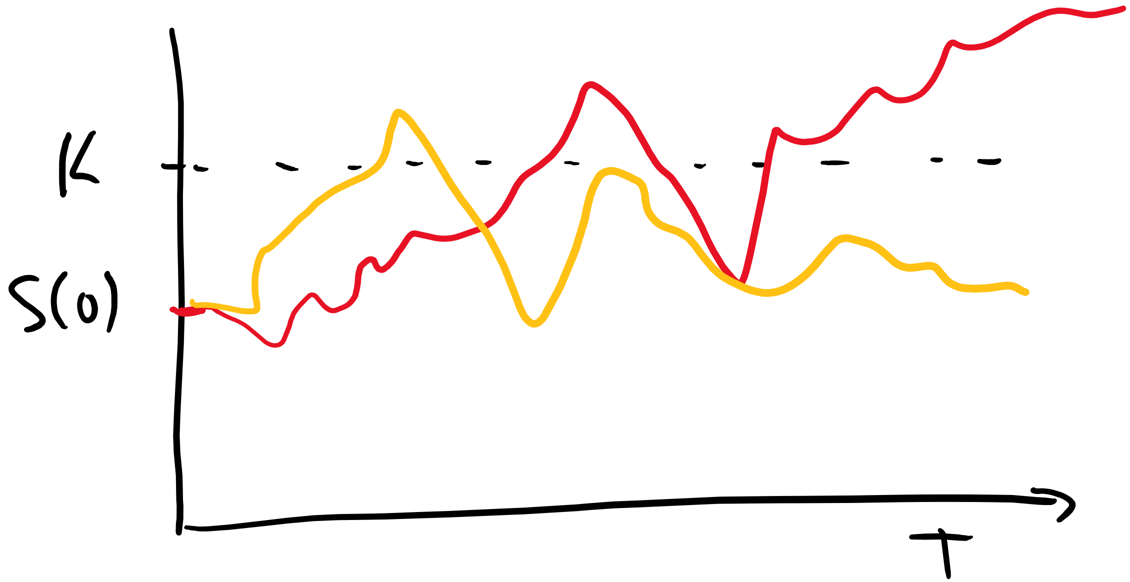
contract function $g(K, S(t))$

Such that at the maturity the buyer of the option has the option to exercise or not the contract

TÉCNICO LISBOA

Options

Exemple



Contract: if at time T, S(t) > K, then the buyer is entitled to by the sotck paying just K. if S(t) < K then the buyer does not exercise his right and therefore he does not buy the stock

Options

Derivative: Claim Whose Payoff is a Function of Another's

- **Warrants**, **Options**
- **Calls** vs. **Puts**
- **American** vs. **European**
- Terms: **Strike Price** *K*, **Time to Maturity** τ

$$\begin{aligned} S_t &= \text{Stock Price} \quad , \quad K = \text{Strike Price} \\ C_t &= \text{Call Price} \quad , \quad P_t = \text{Put Price} \end{aligned}$$

- At Maturity *T*, payoff

$$\begin{aligned} C_T &= \text{Max}\ [\ 0\ ,\ S_T - K\ ] \\ P_T &= \text{Max}\ [\ 0\ ,\ K - S_T\ ] \end{aligned}$$

TÉCNICO LISBOA

Options

Put Options As Insurance

$$
\begin{aligned}
\text{Asset Insured} &= \text{Stock} \\
\text{Current Asset Value} &= \\
\text{Term of Policy} &= \\
\text{Maximum Coverage} &= \\
\text{Deductible} &= \\
\text{Insurance Premium} &=
\end{aligned}
$$

- Differences
  - Early exercise
  - Marketability
  - Dividends

Options

The owner of the option (buyer) exercises his/her right if and only if $S(t) > K$

$$g(K, S(t)) = [S(t) - K] =$$

$$\underbrace{= \max(S(t) - K, 0)}_{\text{EUROPEAN CALL OPTION}}$$